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Local dimension of differential algebraic variety

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Abstract

We consider a relation between local and global characteristics of a differential algebraic variety. We prove that dimension of tangent space for every regular point of an irreducible differential algebraic variety coincides with dimension of the variety. Additionally, we classify tangent spaces at regular points in the case of one derivation.

1 Introduction

We investigate the local behavior of differential algebraic variety. We mostly attend to tangent spaces and relation between their characteristics and characteristics of the variety. We prove two main results. The first one gives an estimate on dimension of associated graded algebra (theorem 11). The second result is a classification of tangent spaces at regular points in the case of one derivation (theorem 16).

In section 2 we briefly recall some definitions. Next section 3 is entirely devoted to an auxiliary technique used further. In subsection 3.1 the notion of height of finitely generated algebra over a field is discussed. In subsection 3.2 we introduce the notion of differential dimension. Differential dimension is defined for differentially finitely generated algebras over a field (section 3.2.1) and for differential algebras of finite type (section 3.2.2). To investigate the local structure of differential algebraic variety we need the notion of tangent space. In section 3.3 we introduce the notion of linear differential space. All tangent spaces appeared are linear differential spaces. So, investigation of such spaces is a preparation for investigation of tangent spaces. Subsection 3.3.1 is devoted to the definition and basic properties. Additionally, a theorem about correspondence between submodules and subspaces is proven in this subsection (theorem 8). Subsection 3.3.2 introduces some useful technique for further applications. Using this machinery, in subsection 3.3.3, we define differential dimension polynomial of linear differential space. Section 4 is devoted to the first main result – theorem 11. This result gives us an estimate on dimension of associated graded algebra. Additionally, we present an example showing that in this estimate the inequality can not be changed to the equality. Next section 5 is devoted to an application of obtained results to tangent spaces. In subsection 5.2 we recall the notion of regular point. We show that differential dimension polynomial of tangent space at regular point coincides with

differential dimension polynomial of the variety (statement 13). The result of section 4 is applied in subsection 5.3 to describe the behavior of points similar to regular ones. In last section 6 we classify tangent spaces at regular points of an irreducible differential algebraic variety in the case of one derivation. The main result is theorem 16. Additionally, we present an example showing that structure of a tangent space depends on the choice of a regular point.

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2 Terms and notation

The word ring means an associative commutative ring with an identity element. All homomorphisms preserve the identity element. A differential ring is a ring with a finitely many pairwise commuting derivations. The set of all derivations will be denoted by Δ. Additionally, we suppose differential rings to be Ritt algebras. The field of fractions of an integral domain A will be denoted by $\mathrm{Qt}(A)$.

3 Preliminaries

In this section we develop an auxiliary technique.

3.1 Dimension

The section is devoted to the notion of the height of finitely generated algebra over a field. Additionally, we shall fix notation and recall the notion of dimension.

Let K be a field, and let B be a finitely generated algebra over K. Let a family $x = (x_1, \ldots, x_n)$ generate the algebra B over K. Define the following subspaces

$$I = <x_1, \ldots, x_n>_K,\ \ I^0 = K$$

and

$$I_n = \sum_{k=0}^{n} I^k.$$

Consider the following function $\chi_x(n) = \dim_K I_n$.

Statement 1. *Using notation above, the following holds:*

1. *The function $\chi_x(n)$ coincides with a polynomial for sufficiently large n.*
2. *If $y = (y_1, \ldots, y_k)$ is another system of generators, then*

$$\deg \chi_x = \deg \chi_y.$$

Proof. Let us show the first statement. The algebra B can be presented as follows

$$K[x_1, \dots, x_n]/\mathfrak{a}.$$

Let some order on monomials of $K[x_1, \dots, x_n]$ be fixed. We suppose that the order preserve degree, for example deglex. Let $g_1, \dots, g_s$ be a Gröbner basis of $\mathfrak{a}$. Then dimension of I_n coincides with the number of all monomials of degree less then of equal to n such that they are not divided by the leading monomials of g_i. Due to lemma [7, chapter 0, sec. 17, lemma 16] it is clear that the number of such monomials coincides with some polynomial.

Let us prove the second statement. Let some other system of generators y be given. The corresponding sequence of subspaces will be denoted by J_n. Then for some m_0 we have the inclusion $J \subseteq I_{m_0}$. Hence

$$J^m \subseteq (I_{m_0})^m = \left(\sum_{k=0}^{m_0} I^k\right)^m \subseteq \sum_{k=0}^{m_0 m} I^k = I_{m_0 m}.$$

Consequently, $\chi_y(m) \leqslant \chi_x(m_0 m)$. In analogue way we have that for some n_0 the inequality $\chi_x(n) \leqslant \chi_y(n_0 n)$ holds. Since both functions are polynomials, $\deg \chi_x = \deg \chi_y$. □

Degree of polynomial χ_x will be called a height of algebra B and will be denoted by $\operatorname{ht} B = \deg \chi_x$. The previous statement guaranty that the notion of height does not depend on the set of generators. Now we shall prove some basic properties of height.

Statement 2. *Let A and B be finitely generated algebras over a field K. Then the following holds:*

1. *If $A \subseteq B$, then* $\operatorname{ht} A \leqslant \operatorname{ht} B$.

2. *If B is a quotient algebra of algebra A, then* $\operatorname{ht} A \geqslant \operatorname{ht} B$.

3. *If $A \subseteq B$ and B is integral over A, then* $\operatorname{ht} A = \operatorname{ht} B$.

4. *The height of polynomial ring in n variables coincides with n, in other words*
$$\operatorname{ht} K[x_1, \dots, x_n] = n.$$

5. *Let some set of algebraically independent elements $y_1, \dots, y_n$ of algebra A be chosen such that algebra A is integral over $K[y_1, \dots, y_n]$, then* $\operatorname{ht} A = n$.

Proof. (1). Let $x = (x_1, \dots, x_n)$ be a family of generators of A. This family can be extended to a set of generators of B. Assume that $x' = x \cup \{ x_{n+1}, \dots, x_r \}$. Let $I = {<}x_1, \dots, x_n{>}$ and $J = {<}x_1, \dots, x_r{>}$. Then $I_n \subseteq J_n$. Consequently, $\chi_x(n) \leqslant \chi_{x'}(n)$. And thus $\deg \chi_x \leqslant \deg \chi_{x'}$.

(2). Let $x = (x_1, \dots, x_n)$ be a family of generators of A. The images of elements of x will be chosen as the generators of B. Let I_n and J_n be corresponding sequences of subspaces in A and B respectively. Then J_n is the

image of I_n under the quotient mapping. Consequently, $\chi_x(n) \geqslant \chi_{\overline{x}}(n)$. So, $\deg \chi_x \geqslant \deg \chi_{\overline{x}}$.

(3). From item (1) it follows that $\operatorname{ht} A \leqslant \operatorname{ht} B$. We just need to show the other inclusion. Let a family $m_1, \ldots, m_r$ generate B as a module over A. Then for some x_{ij}^k we have $m_i m_j = \sum_k x_{ij}^k m_k$. Let us chose a family of generators of A such that the set x_{ij}^k is included to the chosen family of generators. Let denote this family by x and the corresponding sequence of subspaces by I_n. We shall denote by y the following set of generators of B: $x_i m_j$ whenever $x_i \in x$ $1 \leqslant j \leqslant r$. The corresponding sequence of subspaces for y will be denoted by J_n. From the definition we have $J = Im_1 + \ldots + Im_r$. Then

$$J_n = \sum_{k=0}^{n} J^k = \sum_{k=0}^{n} \left(\sum_{i=1}^{r} I m_i \right)^k = \sum_{k=0}^{n} \sum I^k m_{i_1} \ldots m_{i_k} \subseteq$$
$$\subseteq \sum_{k=0}^{n} \sum_{i=1}^{r} I^{2k-1} m_i = \sum_{i=1}^{r} \left(\sum_{k=0}^{n} I^{2k-1} \right) m_i \subseteq \sum_{i=1}^{r} I_{2n-1} m_i.$$

Consequently, $\chi_y(n) \leqslant r\chi_x(2n-1)$. So, $\deg \chi_y \leqslant \deg \chi_x$.

(4). A straightforward calculation show that the number of all monomials in n variables of degree not greater than m is equal to $\binom{n+m}{n}$. The last expression is a polynomial in n variables of degree m.

(5). The statement is an immediate corollary of previous two items. $\square$

Now we shall recall the notion of dimension for a ring A. We use the same notation as in books [1] and [8]. The Krull dimension of algebra A will be denoted by $\dim A$. If A is a local ring with a maximal ideal $\mathfrak{m}$, then $G_{\mathfrak{m}}(A)$ will denote the corresponding associated graded ring, in other words

$$G_{\mathfrak{m}}(A) = \bigoplus_{k=0}^{\infty} \mathfrak{m}^n / \mathfrak{m}^{n+1}.$$

Let

$$P(t) = \sum_{n=0}^{\infty} \dim_{A/\mathfrak{m}} \mathfrak{m}^n / \mathfrak{m}^{n+1}$$

be the Poincaré series of $G_{\mathfrak{m}}(A)$ (see [1, chapter 11, sec. 1]). Order of the pole of $P(t)$ at $t = 1$ we shall denote by $d(A)$. Speaking about height of the algebra $G_{\mathfrak{m}}(A_{\mathfrak{m}})$, we shall consider this ring as the algebra over the field $A/\mathfrak{m}$.

For every local noetherian ring A there is the equality $\dim A = d(A)$ (see [1, chapter 11, sec. 2, th. 11.14]). From the other hand, if A is a finitely generated integral domain over a field, then it is universally catenary (see [8, chapter 5, sec. 14, coroll. 3(2)]). Thus for every maximal ideal $\mathfrak{m}$ of A we have $\dim A = d(A_{\mathfrak{m}})$.

Statement 3. *Let A be a finitely generated algebra over a field K, and let $\mathfrak{m}$ be a maximal ideal of A. Then the following holds:*

1. $\operatorname{ht} A = \dim A$.

2. *If A is an integral domain, then*

$$\operatorname{ht} G_{\mathfrak{m}}(A_{\mathfrak{m}}) = d(A_{\mathfrak{m}}) = \dim A.$$

Proof. (1). Consider the Noether's normalization of A. So, there is a family of algebraically independent elements $y_1, \ldots, y_r$ of A such that A is integral over $K[y_1, \ldots, y_r]$. Then from condition (5) of previous statement it follows that $\operatorname{ht} A = r$. Corollary [8, chapter 5, sec. 14, coroll. 1] guaranties that $\dim A = r$.

(2). Corollary [8, chapter 5, sec. 14, coroll. 3(1)] shows that $d(A_{\mathfrak{m}}) = \dim A$. Corollary [1, chapter 11, sec. 1, coroll. 11.2] guaranties the other equality. □

3.2 Differential dimension

3.2.1 Differentially finitely generated algebras

At present there are two different ways to define differential dimension of differentially finitely generated algebra over a differential field. We shall recall both definitions and show that they gives the same results.

Let K be a differential field with a set of derivations Δ, and let the number of derivations equals m. Consider an algebra A differentially finitely generated over a field K. Let $x = (x_1, \ldots, x_n)$ be a system of differential generators of A, in other words $A = K\{x_1, \ldots, x_n\}$. We shall define the notion of differential height of A. Let $A_k = K[\theta_1 x_1, \ldots, \theta_n x_n \mid \operatorname{ord} \theta_i \leqslant k]$. For every ideal $\mathfrak{a}$ of A the image of the system x in $A/\mathfrak{a}$ will be denoted by $\overline{x}$. Consider the function $\chi_x^A(k) = \dim A_k$.

Statement 4. *Using notation above the following holds:*

1. *The function $\chi_x^A(t)$ is a polynomial for sufficiently large t.*

2. *Let l be degree of the polynomial $\chi_x^A(t)$ and a_k be its coefficient at t^k. Then the numbers l and a_l do not depend on the system x. Moreover, l does not exceed m, and the number $d_l = l!a_l$ is integer. Particulary, the number $d_m = m!a_m$ is integer.*

3. *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the set of all minimal prime (and thus differential) ideals of A. Then*
$$\chi_x^A(k) = \max_{1 \leqslant i \leqslant r} \chi_{\overline{x}}^{A/\mathfrak{p}_i}(k) = \chi_{\overline{x}}^{A/\mathfrak{p}_j}(k)$$
for some j.

4. *If A is an integral domain, then d_m coincides with differential transcendence degree of the fraction field of A over K.*

Proof. Let $\mathfrak{n}$ be the nilradical of A, then its contraction to A_k coincides with the nilradical $\mathfrak{n}_k$. Since $\dim A_k = \dim A_k/\mathfrak{n}_k$, we may suppose that our algebra is reduced. Let contraction of ideal $\mathfrak{p}_i$ to A_k be denoted by $\mathfrak{p}_{ik}$. Then from the

definition of Krull dimension we have $\dim A_k = \max_i \dim A_k/\mathfrak{p}_{ik}$. Since $A/\mathfrak{p}_i$ is an integral domain, theorem [7, chapter 2, sec. 12, th. 6(a)] guaranties that for sufficiently large k the functions

$$\chi_{\overline{x}}^{A/\mathfrak{p}_i}(k) = \dim(A/\mathfrak{p}_i)_k = \dim A_k/\mathfrak{p}_{ik}$$

are polynomials of degree not greater than m. Consequently,

$$\max_{1\leqslant i\leqslant r} \chi_{\overline{x}}^{A/\mathfrak{p}_i}(k)$$

coincides with one of the polynomials $\chi_{\overline{x}}^{A/\mathfrak{p}_j}(k)$ for some j. The last polynomial coincides with $\chi_x(k)$. So, we have proven (1) and (3).

Let us show that degree l and the leading coefficient a_l do not depend on the choice of x. Let $y = (y_1, \ldots, y_s)$ be other system of differential generators. The corresponding filtration will be denoted by A'_k. Then for some k_0 we see that x_i belong to A'_{k_0}. Consequently, we have $A_k \subseteq A'_{k+k_0}$. And thus

$$\chi_x^A(k) \leqslant \chi_y^A(k + k_0).$$

By the similar arguments we have the other inequality

$$\chi_y^A(r) \leqslant \chi_x^A(r + r_0).$$

So, degree and leading coefficient of these polynomials coincide. The fact that l is not greater than m has been proven. Let us show that $l!a_l$ is integer. Indeed, since $\chi_x^A(t)$ is integer for sufficiently large t, the polynomial

$$\Delta\chi_x^A(t) = \chi_x^A(t) - \chi_x^A(t-1).$$

has the same property. Then the number $d_l = \Delta^l\chi_x^A(t) = l!a_l$ is integer. The last item of the statement follows from theorem [7, chapter 2, sec. 12, th. 6(c)]. $\square$

We shall use the notation of the previous statement. The number l is called a type of differential algebra A and denoted by $\operatorname{type} A$. The number d_l is called a typical differential height of A. The number d_m is called a differential height of A and is denoted by $\operatorname{ht}_\Delta A$. This definition is due to J. Kovacic. If A is a local differential ring of finitely generated type over a field (A is a localization of differentially finitely generated algebra), saying about differential height of $G_\mathfrak{m}(A)$, we shall consider this ring as an algebra over the field $A/\mathfrak{m}$.

The following definition appeared in [3, sec. 1, pp. 207-208]. For every differentially finitely generated algebra over a field K consider the function

$$\mu\colon \operatorname{Spec}^\Delta A \times \operatorname{Spec}^\Delta A \to \mathbb{Z} \cup \{\infty\}$$

defined in lemma [3, pp. 208]. Then the maximal value of the function μ will be called a differential Krull type of A and denoted by $\operatorname{type}_K A$. Theorem [3, sec. 2, pp. 208] says that differential Krull type does not exceed m. Consider a longest chain of prime differential ideals $\mathfrak{p}_0 \supseteq \mathfrak{p}_1 \supseteq \ldots \supseteq \mathfrak{p}_k$ such that $\mu(\mathfrak{p}_i, \mathfrak{p}_{i+1}) =$

$\operatorname{type}_K A$. Then the maximum of such k will be called a typical Krull dimension of A. Differential Krull dimension of A is the maximum of k such that $\mathfrak{p}_0 \supseteq \mathfrak{p}_1 \supseteq \ldots \supseteq \mathfrak{p}_k$ and $\mu(\mathfrak{p}_i, \mathfrak{p}_{i+1}) = m$. It is clear, that differential Krull dimension equals zero if and only if differential Krull type is less than m. Differential Krull dimension will be denoted by $\dim_\Delta A$. It should be noted that our definition of Krull dimension a little bit differ from the initial one. Theorem [3, sec. 2, pp. 208] says that if A is an integral domain, then differential Krull dimension coincides with differential transcendence degree of fraction field of A over K. From item (4) of the previous statement we have.

Statement 5. *Let A be a differentially finitely generated integral domain over a field K. Then* $\operatorname{ht}_\Delta A = \dim_\Delta A$.

We shall show that the equality holds in the case of arbitrary differentially finitely generated algebra.

Statement 6. *For any algebra A differentially finitely generated over K there is the equality* $\operatorname{ht}_\Delta A = \dim_\Delta A$.

Proof. From statement 4 it follows that $\operatorname{ht}_\Delta A$ coincides with the maximum of $\operatorname{ht}_\Delta A/\mathfrak{p}_i$, where $\mathfrak{p}_i$ is a minimal prime differential ideal in A. From the other hand, from definition it follows that $\dim_\Delta A$ coincides with the maximum of $\dim_\Delta A/\mathfrak{p}_i$, where $\mathfrak{p}_i$ is a minimal prime differential ideal of A. From previous statement it follows that both these numbers coincide. $\square$

3.2.2 Differential algebras of finite type

Let A be a differential algebra over a field K such that there exists a system of elements $x = (x_1, \ldots, x_n)$ in A with the following property. Algebra A coincides with $K\{x_1, \ldots, x_n\}_\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of $K\{x_1, \ldots, x_n\}$. The system of elements x will be called a system of local generators of A. Let B denote the differential subalgebra in A generated by $x_1, \ldots, x_n$. The maximal ideal of A will be denoted by $\mathfrak{m}$. The images of x_i in $B/\mathfrak{p}$ will be denoted by $\overline{x}_i$. Then the system $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n)$ is a system of differential generators of $B/\mathfrak{p}$. For a given system of generators x we shall define the following rings $B_r = K[\theta_i x_i \mid \operatorname{ord}\theta_i \leqslant r]$ and $(B/\mathfrak{p})_r = K[\theta_i \overline{x}_i \mid \operatorname{ord}\theta_i \leqslant r]$. Then define $\mathfrak{p}_r = B_r \cap \mathfrak{p}$, $A_r = (B_r)_{\mathfrak{p}_r}$, and the maximal ideal of A_r will be denoted by $\mathfrak{m}_r$. Consider the function $\chi_x^A(t) = \dim A_t$.

Statement 7. *Using notation above, the following holds:*

1. *The function $\chi_x^A(t)$ coincides with a polynomial for sufficiently large t.*
2. *Let l be degree of the polynomial $\chi_x^A(t)$, and a_k be its coefficients at t^k. Then numbers l and a_l do not depend on choice of x. Moreover, l is not greater than m, and the number $d_l = l!a_l$ is integer. Particulary, the number $d_m = m!a_m$ is integer.*

3. *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be all minimal prime (and thus differential) ideals of algebra A. Then*
$$\chi_x^A(k) = \max_{1\leqslant i\leqslant r} \chi_{\overline{x}}^{A/\mathfrak{p}_i}(k) = \chi_{\overline{x}}^{A/\mathfrak{p}_j}(k)$$
for some j.

Proof. Let us show the equality $\dim A_t = \dim B_t - \dim(B/p)_t$. Since B can be embedded to A, every minimal prime ideal of A is contained in ideal $\mathfrak{p}$. Let us denote the contraction of $\mathfrak{p}_i$ to B_t by $\mathfrak{p}_{it}$. Then from corollary [8, chapter 14, sec 14.H, coroll. 3] we have

$$\dim A_t = \max_{1\leqslant i\leqslant r}(\dim A_t/\mathfrak{p}_{it}) = \max_{1\leqslant i\leqslant r}(\dim B_t/\mathfrak{p}_{it}) - \dim B_t/\mathfrak{p}_t =$$
$$= \dim B_t - \dim B_t/\mathfrak{p}_t = \dim B_t - \dim(B/\mathfrak{p})_t.$$

So,
$$\chi_x^A(t) = \max_{1\leqslant i\leqslant r} \chi_{\overline{x}}^{A/\mathfrak{p}_i}(t) = \chi_x^B(t) - \chi_{\overline{x}}^{B/\mathfrak{p}}(t).$$

Additionally, the polynomial $\chi_x^A(k)$ has integer values and is of degree not greater than m, because it is a difference of two polynomials of degree not greater than m.

Let $y = (y_1, \ldots, y_s)$ be some other system of local generators of A. The corresponding filtration will be denoted by A'_k. Then for some k_0 we have $A_0 \subseteq A'_{k_0}$, and thus $A_k \subseteq A'_{k+k_0}$. Hence

$$\chi_x^A(k) \leqslant \chi_y^A(k + k_0).$$

Using analogous arguments we have the other inequality

$$\chi_y^A(t) \leqslant \chi_x^A(t + t_0).$$

Therefore degrees and the leading coefficients of these polynomials coincide.

Since polynomial $\chi_x(k)$ has integer values for sufficiently large k, the polynomial $\Delta\chi_x(t) = \chi_x(t) - \chi_x(t-1)$ has integral values too. So, the coefficient $d_l = \Delta^l \chi_x(t) = l! a_l$ is integer, where a_l is the leading coefficient. □

Degree of the polynomial χ_x^A is called a differential type of algebra A. Coefficient d_m is called a differential height of A, and coefficient d_l is called a typical differential height.

3.3 Linear differential spaces

In this section we prepare a basic technique for local theory.

3.3.1 Definition

Let K be a differential field. Concider an affine space K^n. A differential module $K[\Delta]^n$ will be denoted by $\mathcal{L}_n$. For every element $\xi \in \mathcal{L}_n$ we define the mapping $\xi\colon K^n \to K$ by the following rule. Let $\xi = (\xi_1, \ldots, \xi_n)$, where $\xi_i \in K[\Delta]$, and let $x = (x_1, \ldots, x_n) \in K^n$. Then $\xi(x) = \xi_1 x_1 + \ldots + \xi_n x_n$. The elements of module $\mathcal{L}_n$ will be called linear differential functions on K^n and will be denoted by $\mathcal{L}(K^n)$.

Consider an arbitrary differential submodule $N \subseteq \mathcal{L}_n$. For this submodule we define the following set

$$V(N) = \{\, x \in K^n \mid \forall \xi \in N\colon \xi(x) = 0 \,\}.$$

Conversely, for every subset $X \subseteq K^n$ we define a differential submodule

$$I(X) = \{\, \xi \in \mathcal{L}_n \mid \xi|_X = 0 \,\}.$$

The sets of the form $V(N)$ will be called linear differential spaces.

For every natural number n consider the set of differential homomorphisms $\mathrm{Hom}_{K[\Delta]}(K[\Delta]^n, K)$. Then there is the mapping

$$\mathrm{Hom}_{K[\Delta]}(K[\Delta]^n, K) \to K^n$$

by the rule $\xi \mapsto (\xi(e_1), \ldots, \xi(e_n))$, where e_i is a standard basis of $K[\Delta]^n$. Since $K[\Delta]^n$ is free module, the constructed mapping is bijective. Let now $N \subseteq K[\Delta]^n$ be a submodule. Then the set of all homomorphisms of $\mathrm{Hom}_{K[\Delta]}(K[\Delta]^n, K)$ vanishing on N maps to $V(N)$ bijectively. It is clear that these homomorphisms can be identified with $\mathrm{Hom}_{K[\Delta]}(K[\Delta]^n/N, K)$. In other words we have the bijection

$$\mathrm{Hom}_{K[\Delta]}(K[\Delta]^n/N, K) \to V(N)$$

by the rule $\xi \mapsto (\xi(e_1), \ldots, \xi(e_n))$.

Now consider the symmetric algebra on the module $K[\Delta]^n$. So,

$$R_n = S_K(K[\Delta]^n).$$

If $e_1, \ldots, e_n$ are standard free generators of the module $K[\Delta]^n$, then the ring R_n coincides with a differential polynomial ring $K\{e_1, \ldots, e_n\}$. Then we may suppose that module $K[\Delta]^n$ is embedded into R_n. For every submodule $N \subseteq K[\Delta]^n$ we define the ideal $[N] = (N)$. Since $[N]$ is a graded ideal with respect to degree, we have $[N] \cap K[\Delta]^n = N$. The ideal $[N]$ is prime because it is generated by linear differential polynomials. Every differential homomorphism $\xi \in \mathrm{Hom}_{K[\Delta]}(K[\Delta]^n, K)$ gives a differential homomorphism $\xi\colon R_n \to K$. The last correspondence is a bijection between the set of all differential homomorphisms of $\mathrm{Hom}_{K[\Delta]}(K[\Delta]^n, K)$ and the set of all differential rings homomorphisms $R_n \to K$.

Statement 8 (Nullstellensatz)**.** *For every differential submodule $N \subseteq \mathcal{L}_n$ we have $N = I(V(N))$.*

Proof. The inclusion $N \subseteq I(V(N))$ is obvious. Let us show the other one. Consider the algebra R_n and the ideal $[N]$. Let $x \in K[\Delta]^n \setminus N$. Since $[N] \cap K[\Delta]^n = N$, $x \notin [N]$. We know that the ideal $[N]$ is prime and the field K is differentially closed. Therefore there exists a point $a \in K^n$ such that for every $f \in [N]$ we have $f(a) = 0$ and $x(a) \neq 0$. So, x is not in $I(V(N))$. □

Let $V \subseteq K^n$ be a linear differential space. Consider the restriction of functions in $K[\Delta]^n$ to V. The resulting module will be denoted by $\mathcal{L}(V)$ and called the module of linear differential functions on V. From the previous theorem it follows that $\mathcal{L}(V)$ is isomorphic to $K[\Delta]^n/I(V)$. Then the symmetric algebra $S_K(\mathcal{L}(V))$ can be identified with coordinate ring of differential algebraic variety $V \subseteq K^n$.

Let $\varphi_1, \ldots, \varphi_m$ be elements of $K[\Delta]^n$. Then we define a linear differential mapping $\varphi\colon K^n \to K^m$ by the rule: if $x \in K^n$, the coordinates of the point $\varphi(x)$ equal $\varphi_1(x), \ldots, \varphi_m(x)$. Let V and U be linear differential spaces in K^n and K^m respectively. Then the mapping $\varphi\colon V \to U$ is called a linear differential mapping if it coincides with a restriction of some linear differential mapping from K^n to K^m.

3.3.2 Characteristic sets

Let $W = K[\Delta]^n$ be a free differential module, and let its standard basis be denoted by $e_1, \ldots, e_n$. Then the module W is generated by θe_i, where $\theta \in \Theta$. By a ranking of $e_1, \ldots, e_n$ we shall mean a total ordering of the set of all derivatives θe_i that satisfies the two conditions

$$u \leqslant \theta u, \;\; u \leqslant v \Rightarrow \theta u \leqslant \theta v.$$

From statement [7, chapter 0, sec. 17, lemma 15] it follows that ranking exists and every ranking is a well ordering of the set of all derivatives θe_i. A ranking will be said to be orderly if the rank of θe_i is less then that of $\theta' e_j$ whenever $\operatorname{ord}\theta < \operatorname{ord}\theta'$.

Let w be an element of W. Then it is a linear combination of elements θe_i. Let u_w denotes a leading derivative involved in w. So, there appear a pre-order on the set W. Namely, we compare two elements by their leading vectors. Let w' be other element of W. We shall say that w' is reduced with respect to w if w' is free of every derivative of u_w. The set of elements of W will be called reduced if every element of this set is reduced with respect to every other element. The fact that every autoreduced set is finite is a corollary of lemma [7, chapter 0, sec. 17, lemma 15(a)] is

Now we shall define a particular order on the set of all autoreduced sets of W. Let $A = \{f_1, \ldots, f_r\}$ and $B = \{g_1, \ldots, g_s\}$ be autoreduced sets such that their elements are arranged in order of increasing rank. We shall say that rank of A is less than rank of B if

1. There exist $k \in \mathbb{N}$ $k \leqslant r$ and $k \leqslant s$ such that $u_{f_i} = u_{g_i}$ whenever $1 \leqslant i < k$ and $u_{f_k} < u_{g_k}$.

2. Or $r > s$ and $u_{f_i} = u_{g_i}$ whenever $1 \leqslant i \leqslant s$.

It should be noted that if $r = s$ and for every i we have $u_{f_i} = u_{g_i}$ $1 \leqslant i \leqslant s$, then A and B has the same rank.

Statement 9. *In every nonempty set of autoreduced subsets of W there exists an autoreduced set of minimal rank.*

Proof. Let $\mathcal{A}$ be any nonempty set of autoreduced subsets of W. Define by induction an infinite decreasing sequence of subsets of $\mathcal{A}$ by the conditions that $\mathcal{A} = \mathcal{A}_0$ and, for $i > 0$, $\mathcal{A}_i$ is the set of all autoreduced sets $A \in \mathcal{A}_{i-1}$ with $|A| \geqslant i$ such that the i-th lowest element of A is of lowest possible rank. It is obvious that in all elements of $\mathcal{A}_i$ the i-th lowest elements have the same leader v_i. If every $\mathcal{A}_i$ were nonempty, then the leaders v_i would form an infinite sequence of derivatives of e_i such that no v_i is a derivative of any other, and this would contradict lemma [7, chapter 0, sec. 17, lemma 15(a)]. Therefore there is a smallest i such that $\mathcal{A}_i = \emptyset$ and, since $\mathcal{A} = \mathcal{A}_0 \neq \emptyset$, $i > 0$. Any element of $\mathcal{A}_{i-1}$ is clearly an autoreduced subset in $\mathcal{A}$ of lowest rank. $\square$

For every submodule $N \subseteq W$ there exists a minimal autoreduced subset consisting of elements of N. Such autoreduced sets we shall call characteristic sets of N.

3.3.3 Dimension

There are two equivalent methods to define dimension of linear differential space. Let V be a linear differential space, and let $\mathcal{L}(V)$ be the set of all linear differential functions on V. Using the module $\mathcal{L}(V)$ we shall define dimension of V.

The set of elements $y_1, \ldots, y_d$ of differential module M over a field K is called differentially independent if the set θy_i is linearly independent. The maximal number of differentially independent elements of module M is called a differential dimension of M.

The second way of defining differential dimension is the following. Let $y_1, \ldots, y_n$ be differential generators of M. The there is a family of subspaces in M

$$M_k = < \theta_1 y_1, \ldots, \theta_n y_n \mid \operatorname{ord} \theta_i \leqslant k >_K .$$

All subspaces M_k are of finite dimension. Therefore there is a function $\varphi(k) = \dim_K M_k$. Statement [5, chapter I, sec. 4, th.] guaranties that for sufficiently large k the function $\varphi(k)$ coincides with a polynomial of degree not greater that $|\Delta|$. This polynomial is called a differential dimension polynomial. The coefficient at term t^m has the form $a_m/m!$. Then the number a_m is called differential dimension of M. Statement [5, chapter III, sec 2, prop.] says that these both definitions coincide. We shall denote differential dimension of module M by $\dim_{\Delta K} M$. Moreover statement [5, chapter III,sec. 2, lemma] says that dimension has an appropriate behavior. Namely, for every exact sequence of differential nodules $0 \to M' \to M \to M'' \to 0$, there is the equality $\dim_{\Delta K} M =$

$\dim_{\Delta K} M' + \dim_{\Delta K} M''$. Differential dimension of linear differential space is a differential dimension of its module of linear differential functions.

The next our purpose is to derive the information about other coefficients of differential dimension polynomial in the case of one derivation. In the following statement we suppose that we deal with the case of one derivation δ.

Statement 10. *Let M be a differentially finitely generated module over a field K of differential dimension d. Let $m_1, \ldots, m_n$ be a system of differential generators, and let $\varphi(t) = dt + r$ be the differential dimension polynomial calculated by the given set of generators. Then among the elements*

$$m_1, \ldots, m_n$$

there are d differentially independent elements $m_{i_1}, \ldots, m_{i_d}$ such that the quotient module $M/[m_{i_1}, \ldots, m_{i_d}]$ is a vector space of dimension r over K.

Proof. The module M can be presented as follows $M = K[\delta]^n/N$, where standard basis e_i maps to m_i. Let us fix an orderly ranking. And let F be a characteristic set for N. Changing the order of elements m_i we can suppose that $m_1, \ldots, m_d$ are differentially independent and the leaders of elements of F are derivatives of $m_{d+1}, \ldots, m_n$. Assume that leaders have the following form $\theta_{d+1}e_{d+1}, \ldots, \theta_n e_n$. It is clear that in this situation a free term of differential dimension polynomial coincides with number of elements in the following set

$$\{\, \theta e_i \mid d+1 \leqslant i \leqslant n,\, \theta e_i < \theta_i e_i \,\}.$$

But this set form a basis of the quotient space

$$M/[m_1, \ldots, m_d].$$

$\square$

4 Associated graded algebra

Let A be a differential algebra over a field K such that there exists a family of elements $x = (x_1, \ldots, x_n)$ with the following property. The algebra A coincides with $K\{x_1, \ldots, x_n\}_{\mathfrak{p}}$, where $\mathfrak{p}$ is a prime differential ideal of $K\{x_1, \ldots, x_n\}$. Such system of elements x we shall call a system of local generators of A. Let B denote the algebra generated over K by elements $x_1, \ldots, x_n$. The maximal ideal of A will be denoted by $\mathfrak{m}$. The images of x_i in $B/\mathfrak{p}$ will be denoted by $\overline{x}_i$. Then the system $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n)$ is the family of differential generators of $B/\mathfrak{p}$. For our system of generators we define the rings $B_r = K[\theta_i x_i \mid \operatorname{ord}\theta_i \leqslant r]$ and $(B/\mathfrak{p})_r = K[\theta_i \overline{x}_i \mid \operatorname{ord}\theta_i \leqslant r]$. Then we define $\mathfrak{p}_r = B_r \cap \mathfrak{p}$, $A_r = (B_r)_{\mathfrak{p}_r}$, and the maximal ideal of A_r will be denoted by $\mathfrak{m}_r$.

Consider an associated graded algebra

$$G_{\mathfrak{m}}(A) = A/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \ldots \oplus \mathfrak{m}^n/\mathfrak{m}^{n+1} \oplus \ldots$$

It coincides with a direct limit of associated graded algebras $G_{\mathfrak{m}_r}(A_r)$. Moreover, $G_{\mathfrak{m}}(A)$ is differentially finitely generated over $A/\mathfrak{m}$.

Theorem 11. *Let notation be as above. If B is an integral domain, then*

1. $\operatorname{type} G_{\mathfrak{m}}(A) \leqslant \operatorname{type} A$.
2. *If* $\operatorname{type} G_{\mathfrak{m}}(A) = \operatorname{type} A$*, then typical differential height of* $G_{\mathfrak{m}}(A)$ *is not greater than typical differential height of* A.
3. $\dim_\Delta G_{\mathfrak{m}}(A) \leqslant \dim_\Delta B - \dim_\Delta B/\mathfrak{p} = \operatorname{ht}_\Delta A$.

Proof. Corollary [8, chapter 5, sec. 14, coroll. 3(1)] says that

$$\dim B_r = \dim A_r + \dim(B_r/\mathfrak{p}_r).$$

We know that $B_r/\mathfrak{p}_r = (B/\mathfrak{p})_r$. Theorem [1, chapter 11, sec. 2, th. 11.14] with statement 3 guaranty that there is the following sequence of equalities

$$\dim A_r = d(A_r) = d(G_{\mathfrak{m}_r}(A_r)) = \operatorname{ht}(G_{\mathfrak{m}_r}(A_r)).$$

Consider the natural mapping $G_{\mathfrak{m}_r}(A_r) \to G_{\mathfrak{m}}(A)$. Let us denote its image by D'_r. Consider the smallest $A/\mathfrak{m}$ algebra generated by D'_r. In other word, consider $A/\mathfrak{m} \cdot D'_r$. We shall denote this algebra by D_r. If

$$D'_r = \oplus_{k\geqslant 0} D'_{rk},$$

where $D'_{r0} = A_r/\mathfrak{m}_r$, then

$$D_r = \oplus_{k\geqslant 0} D_{rk},$$

where $D_{rk} = A/\mathfrak{m} \cdot D'_{rk}$. Consequently, dimension of D'_{rk} over K coincides with dimension of D_{rk} over $A/\mathfrak{m}$. So,

$$\operatorname{ht} D_r = \operatorname{ht} D'_r \leqslant \operatorname{ht} G_{\mathfrak{m}_r}(A_r).$$

From the construction we have $\delta(D_r) \subseteq D_{r+1}$ for every r and every $\delta \in \Delta$. Moreover,

$$G_{\mathfrak{m}}(A) = \bigcup_{k\geqslant 0} D_k.$$

Let $y = (y_1, \ldots, y_t)$ be a system of differential generators of $G_{\mathfrak{m}}(A)$ over $A/\mathfrak{m}$. Then for some r_0 it follows that for all i we have $y_i \in D_{r_0}$. Therefore

$$G_{\mathfrak{m}}(A)_r \subseteq D_{r+r_0}.$$

Hence we have the following sequence of inequalities

$$\begin{gathered}\dim G_{\mathfrak{m}}(A)_r \leqslant \dim D_{r+r_0} = \operatorname{ht} D_{r+r_0} \leqslant \\ \leqslant \operatorname{ht} G_{\mathfrak{m}_{r+r_0}}(A_{r+r_0}) = \dim B_{r+r_0} - \dim(B/\mathfrak{p})_{r+r_0}.\end{gathered}$$

So, for some r_0 we have that for all r the following holds

$$\dim G_{\mathfrak{m}}(A)_r \leqslant \dim B_{r+r_0} - \dim(B/\mathfrak{p})_{r+r_0} = \dim A_{r+r_0}.$$

Since the functions above are polynomials with integer values, all three results immediately follows from the last inequality and definitions. $\square$

We shall show that in the previous theorem we can not change the inequality to an equality. The following example appeared in work [4]. We just add some remarks to it.

Example 12. Let K be a field with one differential operator and $K<y>$ be a differential extension of K such that y and y' are algebraically independent over K and $y'' = y'/y$. If we let $B = K\{y\}$, then $B_0 = K[y]$ and $B_r = K[y, y'/y^{r-1}]$ if $r > 0$. Clearly By is a prime differential ideal of B, and in fact $B/By = K$. Let $A = B_{By}$. Since $y' = y^{r-1}(y'/y^{r-1}) \in By^{r-1}$ for all $r > 1$, $y' \in \cap_{d\geqslant 0}\mathfrak{m}^d$. Let us show that $[y'] = \cap_{d\geqslant 0}\mathfrak{m}^d$. Indeed, It is easy to see that the ring $A' = A_{By}/[y']$ coincides with $K[y]_{(y)}$, where $y' = 0$. But the ring A' is regular, so the intersection of powers of maximal ideal of A' is zero. Let us note that under the homomorphism $A \to A'$ the powers of maximal ideal of A corresponds to the powers of maximal ideal of A' (because the kernel is contained in the intersection $\cap_{d\geqslant 0}\mathfrak{m}^d$). It is clear that $G_{\mathfrak{m}}(A) = G_{\mathfrak{m}}(A') = K[t]$. So, typical differential height of A is equal to its transcendence degree 2, but typical differential height of associated graded ring is 1.

5 Cotangent spaces

5.1 Definition

Let K be a differential field with the set of derivations Δ. Let B be a differentially finitely generated algebra over K not containing nilpotent elements. Then for such algebra B we can produce a differential algebraic variety as follows. Let $y_1, \ldots, y_n$ be a family of differential generators of B. Then $B = K\{y_1, \ldots, y_n\}/\mathfrak{a}$, where $\mathfrak{a}$ is a radical differential ideal. We define the set $X = V(\mathfrak{a})$ consisting of all common zeros for $\mathfrak{a}$ in K^n. So, B can be identified with a coordinate ring of X is sense of [2]. Let $\mathfrak{m} \subseteq B$ be a maximal differential ideal of B. Since K is differentially closed, $\mathfrak{m} = [y_1 - a_1, \ldots, y_n - a_n]$. So all maximal differential ideals correspond to points of X. If $x \in X$ is a point of the variety, then the corresponding maximal differential ideal will be denoted by $\mathfrak{m}_x$ or simpler by $\mathfrak{m}$, when it is known what point is under consideration.

We shall define cotangent space for a given point x. Consider local ring $A = B_{\mathfrak{m}}$, where $\mathfrak{m}$ corresponds to x. Then the ideal $\mathfrak{m}$ is differentially finitely generated in A. Therefore the quotient module $\mathfrak{m}/\mathfrak{m}^2$ is differentially finitely generated over $A/\mathfrak{m}$. Since K is differentially closed the field $A/\mathfrak{m}$ coincides with K. In other words, module $\mathfrak{m}/\mathfrak{m}^2$ is differentially finitely generated over K. This module will be called a cotangent space for x and denoted by T^*_x.

We shall describe the second construction of cotangent space. Let $\Omega_{A/K}$ be the module of Kähler differentials of A over K. In [6] it is shown that this module is a differential module over A. Then statement [8, chapter 10, sec. 25, th. 58] guaranties that $\Omega_{A/K} \otimes_A A/\mathfrak{m} = \mathfrak{m}/\mathfrak{m}^2$. It is easy to see that the last isomorphism is an isomorphism of differential modules. This definition allows to calculate cotangent spaces in applications. Namely, if the differential ring B

is of the following form

$$K\{y_1,\ldots,y_n\}/[f_1,\ldots,f_s],$$

then module of differentials has the following form

$$\Omega_{B/K} =< dy_1,\ldots,dy_n > /[df_1,\ldots,df_s].$$

Using localization and tensor products, we are able to calculate cotangent space of every point.

Tangent space for x is a linear differential space corresponding to module $\mathfrak{m}/\mathfrak{m}^2$. Explicitly, let $y_1,\ldots,y_n$ be the set of differential generators of $\mathfrak{m}/\mathfrak{m}^2$. Then this module has the following form $\mathfrak{m}/\mathfrak{m}^2 = K[\Delta]^n/N$. The tangent space T_x coincides with $V(N) \subseteq K^n$.

5.2 Regular points of differential spectrum

We shall use the definition of a regular point of differential spectrum from [4]. Whenever a local ring A is given we shall always associate with it the following notation

- $\mathfrak{m}$ the maximal ideal of A.
- $P(A) = S_K(\mathfrak{m}/\mathfrak{m}^2)$ the symmetric algebra over K on vector space $\mathfrak{m}/\mathfrak{m}^2$.
- $G(A) = \oplus_{d\geqslant 0}\mathfrak{m}^d/\mathfrak{m}^{d+1}$ the associated graded algebra of A.
- $\tau_A\colon P(A)\to G(A)$ the unique K-algebra homomorphism that extends the identity mapping of $\mathfrak{m}/\mathfrak{m}^2$.

Let A be a differential algebra over K such that there is a system of elements $x=(x_1,\ldots,x_n)$ in A with the following property. The algebra A coincides with $K\{x_1,\ldots,x_n\}_\mathfrak{p}$, where $\mathfrak{p}$ is a prime differential ideal of $K\{x_1,\ldots,x_n\}$. The K-algebra generated by elements $x_1,\ldots,x_n$ will be denoted by B. The maximal ideal of A will be denoted by $\mathfrak{m}$. For our system of generators we define the ring $B_r = K[\theta_i x_i \mid \operatorname{ord}\theta_i \leqslant r]$. Then we define $\mathfrak{p}_r = B_r\cap\mathfrak{p}$, $A_r = (B_r)_{\mathfrak{p}_r}$, and the maximal ideal of A_r will be denoted by $\mathfrak{m}_r$.

We shall say that the ring A is regular if the following conditions hold:

(**A1**): The mapping $\tau_A\colon P(A)\to G(A)$ is an isomorphism.

(**A2**): There exists a system of local generators for A such that

$$\phi_r\colon A/\mathfrak{m}\underset{A_r}{\otimes}\Omega_{A_r/K}\to A/\mathfrak{m}\underset{A}{\otimes}\Omega_{A/K}$$

is injective for all $r\geqslant 0$.

Let B be a differentially finitely generated algebra over a field K. Let $\mathfrak{p}$ be a prime differential ideal of B. We shall say that $\mathfrak{p}$ is a regular point of differential spectrum if the local ring $B_\mathfrak{p}$ is regular. The set of local generators in condition (**A2**) we shall call the set of regular local generators of A or the set of regular generators for $\mathfrak{p}$ in B.

It should be noted that for every differentially finitely generated integral domain the set of all regular points is open and everywhere dense [4, sec. 5, pp. 228, th.].

For every system of local generators $x = (x_1, \ldots, x_n)$ of A the set $dx = (dx_1, \ldots, dx_n)$ is a system of differential generators of $\Omega_{A/K}$. Therefore their images $\overline{dx} = (\overline{dx}_1, \ldots, \overline{dx}_n)$ are differential generators of

$$A/\mathfrak{m} \underset{A}{\otimes} \Omega_{A/K} = \mathfrak{m}/\mathfrak{m}^2.$$

Define the sequence of modules $M_k \subseteq M = \mathfrak{m}/\mathfrak{m}^2$ as follows

$$M_k = <\theta_i \overline{dx}_i \mid \operatorname{ord}\theta_i \leqslant k>_K .$$

Let $\Omega'_r = A \otimes_{A_r} \Omega_{A_r/K}$. Then it is easy to see that

$$M_r = \phi_r(A/\mathfrak{m} \underset{A}{\otimes} \Omega'_r).$$

Dimension of vector space M_t over K will be denoted by $\chi^M_{dx}(t)$.

Statement 13. *Let notation be as above. Suppose that the local ring A is regular. Let $B \subseteq A$ be a differential subalgebra generated by regular system of generators. The ideal $\mathfrak{p}$ will denote the contraction of $\mathfrak{m}$ to B. Then*

$$\chi^M_{dx}(t) = \chi^B_x(t) = \chi^{B/\mathfrak{p}}_{\overline{x}}(t) + \chi^A_x(t).$$

Proof. Since A is regular, then A is an integral domain [4, sec. 2, prop. 1]. Since finitely generated integral domain B_t is universally catenary, then we have the equality $\chi^B_x(t) = \chi^{B/\mathfrak{p}}_{\overline{x}}(t) + \chi^A_x(t)$ (see [8, chapter 14, sec. 14.H, coroll. 3]).

Since A_t is a regular local ring (see [4, sec. 2, prop. 1]), then

$$\operatorname{tr}\deg_K \operatorname{Qt}(A) = \operatorname{rk}\Omega_{A_t/K}$$

(see [4, sec. 1, lemma 4(2)]). Consequently, we have the sequence of equalities

$$\dim B_t = \operatorname{tr}\deg_K A = \operatorname{rk}\Omega_{A_t/K} = \dim_K(A/\mathfrak{m} \underset{A_t}{\otimes} \Omega_{A_t/K}).$$

Since the mapping ϕ_r is an isomorphism onto its image, then

$$\dim_K(A/\mathfrak{m} \underset{A_t}{\otimes} \Omega_{A_t/K}) = \dim_K M_t.$$

We have $\dim B_t = \dim_K M_k$. So, $\chi^B_x(t) = \chi^M_{dx}(t)$. □

Corollary 14. *Let B be a differentially finitely generated integral domain over a field K. And let $\mathfrak{p}$ be a regular point of differential spectrum of B. Define $A = B_\mathfrak{p}$ and $\mathfrak{m}$ is its maximal ideal. Then*

$$\dim_{\Delta K} \mathfrak{m}/\mathfrak{m}^2 = \dim_\Delta B - \dim_\Delta B/\mathfrak{p}.$$

5.3 Dimension

Let B be a differentially finitely generated algebra over a field K, and let $\mathfrak{p}$ be a prime differential ideal of B. Consider the ring $A_{\mathfrak{p}}$. Its maximal ideal will be denoted by $\mathfrak{m}$. We are interested in local rings satisfying condition (**A1**) from the definition of regular point of differential spectrum.

Theorem 15. *Let B be a differentially finitely generated integral domain over K and let $\mathfrak{p}$ be a point of differential spectrum satisfying condition* (**A1**)*. Let us denote the residue field of $\mathfrak{p}$ by L, the fraction field of B by F, and $A = B_{\mathfrak{p}}$ be a local ring with a maximal ideal $\mathfrak{m}$. Then the following holds*

$$\dim_{\triangle} \mathfrak{m}/\mathfrak{m}^2 = \dim_{\triangle} B - \dim_{\triangle} B/\mathfrak{p}$$

or using the language of the fields

$$\dim_{\triangle} \mathfrak{m}/\mathfrak{m}^2 = \operatorname{tr}\deg^{\triangle}{}_K F - \operatorname{tr}\deg^{\triangle}{}_K L$$

Proof. The inequality

$$\dim_{\triangle} \mathfrak{m}/\mathfrak{m}^2 \geqslant \dim_{\triangle} B - \dim_{\triangle} B/\mathfrak{p}$$

was proven in theorem [6, sec. 4, pp. 96]. We shall show the other one. Indeed, for every given point $\mathfrak{m}$ we know that

$$P(A) = G(A).$$

Then, from one hand, differential dimension of module $\mathfrak{m}/\mathfrak{m}^2$ coincides with differential dimension of algebra $P(A) = S_K(\mathfrak{m}/\mathfrak{m}^2)$. From the other hand, from theorem 11 it follows that differential dimension of $G(A)$ is not greater than

$$\dim_{\triangle} B - \dim_{\triangle} B/\mathfrak{p},$$

Q.E.D. $\square$

The geometrical meaning of the previous theorem is the following. For every point of differential algebraic variety satisfying condition (**A1**) differential dimension of tangent space coincides with differential dimension of the variety. If we combine this result with theorem [4, sec. 5, pp. 228] we obtain that the set of all such points contains an open everywhere dense set.

6 Classification

The last attainment of our work is a classification of tangent spaces at regular points of differential algebraic varieties in the case of one derivation. The detailed theory of differential algebraic varieties is given in [2]. We shall borrow terms and notation from this work.

Let K be a differentially closed field with the subfield of constants C. Let X be an irreducible differential algebraic variety over K. Let B be a coordinate ring for X.

Theorem 16. *Let X be an irreducible differential algebraic variety. Let $x = (x_1, \dots, x_n)$ be coordinates on X. So,*

$$B = K\{X\} = K\{x_1, \dots, x_n\}/I(X).$$

Assume that a point $y \in X$ is regular in coordinates $x_1, \dots, x_n$, and let differential dimension polynomial has the following form $\chi^B_x(t) = dt + r$. Then the tangent space at y is of the form $T_y = K^d \times C^k$, where d coincides with differential dimension of X and k is less then or equal to r.

Proof. Let $\mathfrak{m}$ be the maximal ideal of a point $y \in X$. Consider the cotangent space $T^*_y = \mathfrak{m}/\mathfrak{m}^2$. This module is a left module over the ring of differential operators $K[\delta]$. The last ring is left and right euclidian [9, chapter 2, sec. 1, lemma 2.1]. Since $\mathfrak{m}/\mathfrak{m}^2$ is finitely generated module over $K[\delta]$, it is isomorphic to

$$K[\delta]^{\oplus n} \oplus K[\delta]/p_1K[\delta] \oplus \dots \oplus K[\delta]/p_sK[\delta].$$

We shall denote the submodule $K[\delta]/p_1K[\delta] \oplus \dots \oplus K[\delta]/p_sK[\delta]$ by V. Then V is a vector space of finite dimension over K. We shall denote its dimension by k. Let $e_1, \dots, e_k$ be a basis of V over K. Then

$$\delta(e_1, \dots, e_k) = (e_1, \dots, e_k)C$$

For some matrix $C \in M_k(K)$. Let us change the basis using a matrix B as follows

$$(e'_1, \dots, e'_k) = (e_1, \dots, e_k)B.$$

So,

$$\delta(e'_1, \dots, e'_k) = (e'_1, \dots, e'_k)C'$$

Then the matrix C' coincides with $B^{-1}CB+B^{-1}\delta B$. We shall solve the equation $C' = 0$. The equation is equivalent to $\delta B = -CB$. Statement [9, chapter 1, sec. 3, prop. 1.20] says that there exists a desired nondegenerated matrix B with coefficients in some Picard-Vessiot extension. But the field K is differentially closed, therefore it contains all Picard-Vessiot extensions. So, the matrix B belongs to $GL_k(K)$. Using vectors $e'_1, \dots, e'_k$ as a basis, we have

$$V = (K[\delta]/\delta K[\delta])^{\oplus k}.$$

So, the module $\mathfrak{m}/\mathfrak{m}^2$ is isomorphic to

$$K[\delta]^{\oplus n} \oplus (K[\delta]/\delta K[\delta])^{\oplus k}.$$

By the definition it is clear that $n = \dim_\Delta \mathfrak{m}/\mathfrak{m}^2$. And from corollary 14 it follows that $n = \dim_\Delta B = \dim X$. The corresponding tangent space has the form $K^d \times C^k$.

We shall show the estimate for k. Statement 13 says that differential dimension polynomial of $\mathfrak{m}/\mathfrak{m}^2$ calculated in induced coordinates coincides with $dt + r$. Submodule V intersects any free submodule T of T^*_y by zero. From statement 10 it follows that there is a free submodule T in T^*_y such that T^*_y/T has dimension r. But V can be embedded to T^*_y/T, and therefore $k = \dim_K V \leqslant \dim_K T^*_y/T = r$ □

Example 17. The goal of this example is to show that coefficient k from the previous statement depends on the choice of a regular point and can take any value from 0 to a value of free term of differential dimension polynomial.

Let K be a differential field with a subfield of constants C. Consider the ring of differential polynomials $K\{z,y\}$ and the ideal $[zy'-y]$. Let us note that the ring $K\{z,y\}_z/[zy'-y]$ has no zero divisors. From criterion [4, sec. 3, pp. 219, th.] it follows that points (z,y) such that $z\neq 0$ are regular points of the variety X given by the equation $zy'-y=0$. Let us note that there is a unique point with condition $z=0$. This point is $(0,0)$.

Consider tangent spaces in all points except $(0,0)$. We shall show that for every point (z_0,y_0) such that $z_0\neq 0, y_0'\neq 0$ the tangent space is K, and for other points (conditions are $z_0\neq 0, y_0'=0$) the tangent space is $K\times C$. Indeed, cotangent space at point $\mathfrak{m}=[z-z_0,y-y_0]$ has the following form

$$\mathfrak{m}/\mathfrak{m}^2=\Omega_{A/K}\otimes_A A/\mathfrak{m},$$

where A is local ring corresponding to the point. Then

$$\mathfrak{m}/\mathfrak{m}^2=<dz,dy>/[z_0dy'+y_0'dz-dy].$$

The last module can be presented as

$$K[\delta]\oplus K[\delta]/(y_0',-1+z_0\delta).$$

Note if $y_0'=0$, then cotangent space is $K[\delta]\oplus K[\delta]/\delta K[\delta]$ otherwise $K[\delta]$.

We shall show the examples of these two possibilities. For the point $(1,0)$ tangent space is specified by the equation $dy'-dy=0$. Let γ be a nonzero solution of the last equation. Then the set of all points $(z,c\gamma)$, where $z\in K$ and $c\in C$, is the tangent space at $(1,0)$ and is isomorphic to $K\times C$. Consider the point (t,t), where $t\in K$ such that $t'=1$. Then the tangent space is specified by the equation $tdy'+dz-dy=0$. Hence the set of all points $(y-ty',y)$, where $y\in K$, is the tangent space at point (t,t) and is isomorphic to K.

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